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Exact Expressions for Row Correlation Functions in the Isotropic $d = 2$ Ising Model

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We present exact explicit expressions for the row spin–spin correlation functions $\langle\sigma_{00}\sigma_{n0}\rangle$ in the isotropic $d=2$ Ising model, in terms of elliptic integrals, for $n \leqslant 5$. We also give a general structural formula for $\langle\sigma_{00}\sigma_{n0}\rangle$.

KEY WORDS: Ising model; solvable models; correlation functions; series expansions.

The $d=2$ Ising model continues to yield new insights into the collective behavior of many-particle systems. This model is especially interesting because one can calculate many quantities of physical interest in exact, analytic form.[(1–5),2] Among these are the (static) spin–spin correlation functions $\langle\sigma_{00}\sigma_{mn}\rangle$. Explicit calculations of these have only been published for $\langle\sigma_{00}\sigma_{01}\rangle$, $\langle\sigma_{00}\sigma_{10}\rangle$ and $\langle\sigma_{00}\sigma_{11}\rangle$.[(1–5)] Recently,[(6)] we found a general structural formula for $S_n = \langle\sigma_{00}\sigma_{nn}\rangle$ involving a homogeneous polynomial, of degree n, in the complete elliptic integrals $E(k_{\gtrless})$ and $K(k_{\gtrless})$, with calculable coefficients comprised of polynomials in $k^2_{\gtrless}$ (for the notation, see Reference 6 or below). We also calculated explicit expressions for S_n with $2 \leqslant n \leqslant 6$.

Here we give a general structural formula for the row, or equivalently, column, correlation functions of the isotropic $d=2$ Ising model, defined by the Hamiltonian

$$H = -\sum_{j,k \in \mathbb{Z}^2} [J_1\sigma_{j,k}\sigma_{j+1,k} + J_2\sigma_{j,k}\sigma_{j,k+1}] \tag{1}$$

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[2] There is a vast literature on the $d=2$ Ising model; we have cited only the few papers which are directly relevant to our calculations. For a detailed discussion of the model and further references, see Reference 5.

with $\sigma_{j,k} = \pm 1, \in \mathbb{Z}_2$ and $J_1 = J_2$. We have also calculated $\langle \sigma_{00}\sigma_{n0} \rangle$ explicitly for $n \leqslant 6$ and present the results for $n \leqslant 5$ here. We define the elliptic moduli

$$k_> = \sinh^2 2\beta J \tag{2}$$

applicable for $T > T_c$ and

$$k_< = k_>^{-1} \tag{3}$$

applicable for $T < T_c$, with $\beta = (k_B T)^{-1}$ and T_c defined by

$$k_>(\beta_c) = k_<(\beta_c) = 1 \tag{4}$$

Thus $0 \leqslant k_{\gtrless} \leqslant 1$. The correlation function $R_n = \langle \sigma_{00}\sigma_{n0} \rangle$ can be formally expressed as a Toeplitz determinant[4]:

$$R_n = \begin{vmatrix} a_0 & a_{-1} & \cdots & a_{1-n} \\ a_1 & a_0 & & \\ \vdots & & \ddots & \\ a_{n-1} & & & a_0 \end{vmatrix} \tag{5}$$

where

$$\begin{aligned} a_r = {} & z_1 \delta_{r,0} + (1 - z_1^2) \\ & \times [z_1(1 + z_1^2) F_{0,r} + (z_2^2 - 1) F_{0,1+r} + 2z_1 z_2 F_{1,r}] \end{aligned} \tag{6}$$

with

$$z_i = \tanh \beta J_i \tag{7}$$

$$F_{m_1,m_2} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} d\varphi_1 \int_{-\pi}^{\pi} d\varphi_2 \frac{e^{i(m_1\varphi_1 + m_2\varphi_2)}}{\Delta(\varphi_1, \varphi_2)} \tag{8}$$

and

$$\begin{aligned} \Delta(\varphi_1, \varphi_2) = {} & (1 + z_1^2)(1 + z_2^2) - 2z_2(1 - z_1^2) \cos \varphi_1 \\ & - 2z_1(1 - z_2^2) \cos \varphi_1 \end{aligned} \tag{9}$$

The matrix element a_r is more complicated, even for the isotropic case, than the analogous quantity in the diagonal case, which has the form

$$b_r = 4z_1 z_2 F_{r,r} - (1 - z_1^2)(1 - z_2^2) F_{r+1,r+1} \tag{10}$$

involving only the diagonal functions $F_{m,m}$. This difference gives rise to the substantially more complicated structure exhibited by R_n, even in the isotropic case, as compared with S_n.

We proceed to give the general structural formulas which we have found for the R_n as explicit polynomials in the complete elliptic integrals of the first and second kind. Denote

$$R_{n,\pm} = \langle \sigma_{0,0}\sigma_{n,0} \rangle_{T>T_c,T<T_c} \tag{11}$$

The general form of $R_{n,\pm}$ depends in part on whether n is even or odd, in contrast to the diagonal correlation functions $S_{n,\pm}$.

Specifically, the forms are

$$n \text{ even:}\quad R_{n,\pm} = D_n k^{-q_n} \sum_{l=0}^{n/2} \pi^{-2l} \sum_{r=0}^{2l} \mathscr{R}^{(n,\pm)}_{2l-r,r}(k) \times (k-1)^r E(k)^{2l-r} K(k)^r \tag{12}$$

$$n \text{ odd:}\quad \begin{Bmatrix} R_{n,+} \\ R_{n,-} \end{Bmatrix} = [\operatorname{sgn}(J)]^n D_n k^{-q_n} \begin{Bmatrix} (1+k_>^{-1})^{1/2} \\ (1+k_<)^{1/2} \end{Bmatrix} \sum_{l=0}^{n} \begin{Bmatrix} 1 \\ (-1)^{n-l+1} \end{Bmatrix} \pi^{-l} \times \sum_{r=0}^{l} \mathscr{R}^{(n)}_{l-r,r}(k)(k-1)^{r+(1/2)[1-(-1)^l]\delta_{r,0}} \times E(k)^{l-r} K(k)^r \tag{13}$$

Here $k = k_>$ ($k_<$) for $T > T_c$ ($T < T_c$); D_n is an inverse integer extracted for convenience; q_n is a positive semidefinite integer; $\mathscr{R}^{(n,\text{even},\pm)}_{2l-r,r}(k_{>,<})$ and $\mathscr{R}^{(n,\text{odd})}_{l-r,r}(k)$ are polynomials in $k_>$ or $k_<$; and $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds, respectively. The formulas (12) and (13) are inferred from the Toeplitz determinant (5), using the properties of the a_r in (6). Specifically, one uses the recursion relation $(1+z^2)^2 F_{m,n} = z(1-z^2)[F_{m+1,n} + F_{m-1,n} + F_{m,n+1} + F_{m,n-1}] + \delta_{m,0}\delta_{n,0}$, where $z_1 = z_2 \equiv z$, to obtain the nondiagonal $F_{m,n}$ from the diagonal ones. The latter are given by $F_{m,m} = [2\pi z(1-z^2)]^{-1} Q_{m-(1/2)}(w)$, where $w = (1+z^2)^4/[8z^2(1-z^2)^2] - 1$, and $Q_\nu(w)$ is the (singular) Legendre function of the second kind. This function has the form $Q_{m-(1/2)}(w) = \alpha_{m,E,\pm} E(k) + \alpha_{m,K,\pm} K(k)$, where the α coefficients are algebraic functions of k and $k = k_>$ or $k_<$, depending on whether $T > T_c$ or $T < T_c$. It follows that $F_{m,n}$ has the form $F_{m,n} = c_{m,n,E,\pm} E(k) + c_{m,n,K,\pm} K(k) + c_{m,n,1,\pm}$, where the c coefficients are again algebraic functions of k (and $c_{m,n,1,\pm} = 0$ if $m = n$). This property yields the general hierarchical form in (13); further cancellations produce the alternative-level structure in (12) for the case of even n. Several interesting features of the results are worth noting:

(1) Even *n*. (a) $R_{n,\pm}$ consists of a hierarchy of polynomials in $E(k)$ and $K(k)$ which are homogeneous, of even degree $2l$. There are no levels of odd degree, in contrast with the case of $R_{n,\pm}$ with n odd. The full hierarchy consists of all of the levels $2l = 0, 2, \ldots, n$.

(b) Each term $E(k)^{2l-r} K(k)^r$ in one of the homogeneous polynomials of a given $2l$th level involves a coefficient function, $\mathscr{R}^{(n,+)}_{2l-r,r}(k_>)$ or $\mathscr{R}^{(n,-)}_{2l-r,r}(k_<)$ for $T > T_c$ or $T < T_c$, respectively, which is itself a polynomial in $k_{\gtrless}$.

(2) Odd *n*. (a) The odd-n row correlation functions have a square root prefactor. They again involve a hierarchy of homogeneous polynomials in $E(k)$ and $K(k)$, but the hierarchy consists of all l levels from $l = 0$ to $l = n$ and not just the even ones.

(b) As indicated in the notation, the same coefficient polynomial, $\mathscr{R}^{(n)}_{l-r,r}(x)$, enters for $T > T_c$ (with $x = k_>$) and for $T < T_c$ (with $x = k_<$), although there are overall sign changes associated with the $(-1)^{n-l+1}$ factor for $T < T_c$. This differs from the even-n case, where $\mathscr{R}^{(n,+)}_{2l-r,r}(x)$ and $\mathscr{R}^{(n,-)}_{2l-r,r}(x)$ are distinct functions of x.

(3) Comparing these results for the row or column correlation functions with the general structural formula which we recently found[6] for the diagonal correlation functions, $S_{n,\pm}$, the most striking contrast is the greater complexity of the $R_{n,\pm}$, even for the isotropic case studied here. This is particularly evident in the hierarchical level structure of the $R_{n,\pm}$. In this terminology, $S_{n,\pm}$ involves only one l level, namely, the top one, $l = n$. A related difference is that the $\mathscr{P}^{(n,\pm)}_{n-r}$ are even functions of $k_{\gtrless}$ while $\mathscr{R}^{(n\,\text{even},\pm)}_{2l-r,r}$ and $\mathscr{R}^{(n\,\text{odd})}_{l-r,r}$ contain odd as well as even powers of $k_{\gtrless}$.

(4) Another general structural feature of $R_{n\,\text{odd},\pm}$ is that it does not involve any pure $E(k)$ term:

$$\mathscr{R}^{(n\,\text{odd})}_{1,0} = 0 \tag{14}$$

We proceed to derive several relations among the coefficient functions. As $T \to \infty$, $R_{n,+} \to \delta_{n,0}$, which implies that for the nontrivial case $n \neq 0$,

$$n \text{ even:} \quad \sum_{l=0}^{n/2} 2^{-2l} \sum_{r=0}^{2l} (-1)^r \mathscr{R}^{(n,+)}_{2l-r,r}(k_> = 0) = 0 \tag{15}$$

$$n \text{ odd:} \quad \sum_{l=0}^{n} (-2)^{-l} \sum_{r=0}^{l} (-1)^r \mathscr{R}^{(n)}_{l-r,r}(k_> = 0) = 0 \tag{16}$$

As $T \to 0$, $R_{n,-} \to [\text{sgn}(J)]^n$, which implies (again, in the nontrivial case $n \neq 0$) that for n even:

$$\sum_{l=0}^{n/2} 2^{-2l} \sum_{r=0}^{2l} (-1)^r \mathscr{R}^{(n,-)}_{2l-r,r}(k_< = 0) = 0 \tag{17}$$

For n odd the condition implied is equivalent to (16).

Finally, as $T \to T_c$, since $\langle \sigma_{00}\sigma_{mn} \rangle$ is a continuous function of T, $R_{n,-}(T_c) = R_{n,+}(T_c)$. This is automatic for odd n, as is evident from (13). For even n, it implies that

$$n \text{ even:} \quad \mathscr{R}_{2l,0}^{(n,+)}(k_> = 1) = \mathscr{R}_{2l,0}^{(n,-)}(k_< = 1) \tag{18}$$

As indicated, this equality holds separately at each level of the hierarchy. Further, writing

$$\mathscr{R}_{2l,0}^{(n,\pm)} = \sum_{j=0}^{j_{\max}(n,2l)} c_{2l,j}^{(n,\pm)} k_{\gtrless}^{j} \tag{19}$$

we find that

$$c_{2l,j}^{(n,+)} = c_{j_{\max}(n,2l)-j}^{(n,-)} \tag{20}$$

so that the equality equivalent to (18) and (19),

$$\sum_{j=0}^{j_{\max}(n,2l)} c_{2l,j}^{(n,+)} = \sum_{j=0}^{j_{\max}(n,2l)} c_{2l,j}^{(n,-)} \tag{21}$$

is met in the special manner implied by (20); i.e., there is a one-to-one equality between the individual terms in the left-hand side of (21) and the reordered terms in the right-hand side of (21).

We list below our results for the coefficient functions $\mathscr{R}_{2l-r,r}^{(n\,\text{even},\pm)}(k_{\gtrless})$ and $\mathscr{R}_{l-r,r}^{(n\,\text{odd})}(k)$ for $n = 1$ through 3. As stated above, for n odd, $\mathscr{R}_{l-r,r}^{(n\,\text{odd})}$ is the same function of $k_>$ for $T > T_c$ as it is of $k_<$ for $T < T_c$; accordingly, we use the symbol k to denote either $k_>$ or $k_<$. The values of the constants D_n appearing in (12) and (13) are: $D_1 = D_2 = D_3 = 1$, $D_4 = 3^{-2}$, and $D_5 = 3^{-4}$; the q_n are listed in Table I. Results for $n = 4, 5$ are given in the Appendix.

Table I. Values of the Powers q_n

n	q_n
1	0
2	1
3	2
4	4
5	6

***n* = 1:**

$$\mathscr{R}^{(1)}_{1,0} = 0$$
$$\mathscr{R}^{(1)}_{0,1} = 1$$
$$\mathscr{R}^{(1)}_{0,0} = 2^{-1}$$

***n* = 2:**

$T > T_c$:
$$\mathscr{R}^{(2,+)}_{2,0} = -2^2$$
$$\mathscr{R}^{(2,+)}_{1,1} = 0$$
$$\mathscr{R}^{(2,+)}_{0,2} = 2(k_> + 1)$$
$$\mathscr{R}^{(2,+)}_{0,0} = 2^{-1}(k_> + 1)$$

$T < T_c$:
$$\mathscr{R}^{(2,-)}_{2,0} = -2^2$$
$$\mathscr{R}^{(2,-)}_{1,1} = -2^3(k_> + 1)$$
$$\mathscr{R}^{(2,-)}_{0,2} = -2(k_> + 1)(k_> + 2)$$
$$\mathscr{R}^{(2,-)}_{0,0} = 2^{-1}k_>(k_> + 1)$$

***n* = 3:**

$$\mathscr{R}^{(3)}_{3,0} = -2^4$$
$$\mathscr{R}^{(3)}_{2,1} = -2^2(k^2 - 6k - 11)$$
$$\mathscr{R}^{(3)}_{1,2} = 2^3(k + 1)(3k + 5)$$
$$\mathscr{R}^{(3)}_{0,3} = 2^2(k + 1)^2(k + 3)$$
$$\mathscr{R}^{(3)}_{2,0} = -2(k^2 + 6k + 1)$$
$$\mathscr{R}^{(3)}_{1,1} = -2^2(k + 1)(3k + 1)$$
$$\mathscr{R}^{(3)}_{0,2} = -2(k + 1)^3$$
$$\mathscr{R}^{(3)}_{1,0} = 0$$
$$\mathscr{R}^{(3)}_{0,1} = -k(k + 1)^2$$
$$\mathscr{R}^{(3)}_{0,0} = 2^{-1}k(k + 1)^2$$

As would be expected, it is much easier to calculate the $\langle\sigma_{00}\sigma_{mn}\rangle$, and in particular, R_n, at the special point $T = T_c$ than to calculate these correlation functions in general. Since we have computed the full R_n

themselves, we list below the critical values (besides the $n = 1$ value, which is well known,[2] the $n = 2$ value has been given before[7,8]):

$$\bar{R}_{n,c} = \frac{R_n(T = T_c)}{[\mathrm{sgn}(J)]^n} \tag{22}$$

Then

$$\bar{R}_{1,c} = 2^{-1/2} \simeq 0.707 \tag{23a}$$

$$\bar{R}_{2,c} = 1 - \frac{2}{\pi^2} = \left(1 - \frac{2}{\pi}\right)\left(1 + \frac{2}{\pi}\right) \simeq 0.595 \tag{23b}$$

$$\bar{R}_{3,c} = 2^{3/2}\left(1 - \frac{8}{\pi^2}\right) \simeq 0.536 \tag{23c}$$

$$\begin{aligned}\bar{R}_{4,c} &= 2^4\left(1 - \frac{2^4 \cdot 7}{3^2\pi^2} + \frac{2^8}{3^2\pi^4}\right)\\ &= 2^4\left(1 - \frac{2^2}{3\pi} - \frac{2^4}{3\pi^2}\right)\left(1 + \frac{2^2}{3\pi} - \frac{2^4}{3\pi^2}\right) \simeq 0.498\end{aligned} \tag{23d}$$

$$\begin{aligned}\bar{R}_{5,c} &= 2^{15/2}\left[1 - \frac{2^3 \cdot 19}{3^2\pi^2} + \frac{2^9 \cdot 11}{3^4\pi^4}\right]\\ &= 2^{15/2}\left(1 - \frac{8}{3\pi}\right)\left(1 + \frac{8}{3\pi}\right)\left(1 - \frac{88}{3^2\pi^2}\right) \simeq 0.471\end{aligned} \tag{23e}$$

$$\begin{aligned}\bar{R}_{6,c} &= 2^{12}\left[1 - \frac{2^2 \cdot 13 \cdot 31}{3 \cdot 5^2\pi^2} + \frac{2^{10} \cdot 7 \cdot 13}{3^3 \cdot 5^2 \cdot \pi^4} - \frac{2^{22}}{3^6 \cdot 5^2\pi^6}\right]\\ &= 2^{12}\left(1 - \frac{2 \cdot 13}{3 \cdot 5\pi} - \frac{2^5 \cdot 13}{3^2 \cdot 5\pi^2} + \frac{2^{11}}{3^3 \cdot 5\pi^3}\right)\\ &\quad \times \left(1 + \frac{2 \cdot 13}{3 \cdot 5\pi} - \frac{2^5 \cdot 13}{3^2 \cdot 5\pi^2} - \frac{2^{11}}{3^3 \cdot 5\pi^3}\right) \simeq 0.450\end{aligned} \tag{23f}$$

In general,

$$\bar{R}_{n,c} = \sum_{l=0}^{[n/2]} r_{n,l}\pi^{-2l} \tag{24}$$

with rational coefficients which we denote $r_{n,l}$. The occurrence of only even powers of π^{-1} in the sums is obvious for even n since, according to our general formula (12) the hierarchy contains only even-l levels. For odd n, although the hierarchy contains all l levels from $l = 0$ to $l = n$, the odd-l

levels vanish at $T=T_c$ because of the Kronecker $\delta_{r,0}$ term in the factor $(k-1)^{r+(1/2)[1-(-1)^l]\delta_{r,0}}$ in (13). In all cases, only the pure $E(k)^l$ term in each of the even-l levels contributes; all $E(k)^{l-r}K(k)^r$ with $r \neq 0$ have vanishing coefficients (which also annihilate the logarithmic divergence in $K(k)$ as $k \to 1$). The structure (24) may be contrasted with the well-known result for $\langle \sigma_{00}\sigma_{nn} \rangle_{T_c}$ which is a simple monomial $\propto \pi^{-n}$ [and, like the full function $\langle \sigma_{00}\sigma_{nn} \rangle$ itself, is independent of sgn(J)].

Much of the value of solvable models derives from the fact that one can calculate exact analytic expressions for quantities of physical interest, without having to resort to approximation methods. In this spirit we have calculated explicit expressions, in terms of the complete elliptic integrals $E(k)$ and $K(k)$, for the spin–spin correlation functions $\langle \sigma_{00}\sigma_{n0} \rangle$ of the isotropic $d=2$ Ising model for $n \leqslant 6$, and have presented the results for $n \leqslant 5$ here. We have also found an interesting general structural formula for these correlation functions.

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APPENDIX

We list below the $\mathscr{R}_{l-r,r}^{(n\,\mathrm{even},\pm)}(k_{\gtrless})$ and $\mathscr{R}_{l-r,r}^{(n\,\mathrm{odd})}(k)$ (where $k=k_>$ or $k_<$) for $n=4$ and 5. We have also calculated R_6 but the results are too lengthy to present here. They are, of course, available on request.

$n = 4$:

$T > T_c$:

$$\mathscr{R}_{4,0}^{(4,+)} = 2^4(k_>^4 - 84k_>^3 + 86k_>^3 + 204k_> + 49)$$

$$\mathscr{R}_{3,1}^{(4,+)} = -2^8(k_> + 1)(3k_> + 1)(k_>^2 - 6k_> - 11)$$

$$\mathscr{R}_{2,2}^{(4,+)} = -2^6(k_> + 1)(2k_>^4 - 63k_>^3 - 227k_>^2 - 229k_> - 59)$$

$$\mathscr{R}_{1,3}^{(4,+)} = 2^6(k_> + 1)^2(3k_> + 5)(7k_>^2 + 18k_> + 7)$$

$$\mathscr{R}_{0,4}^{(4,+)} = 2^4(k_> + 1)^3(9k_>^3 + 45k_>^2 + 75k_> + 31)$$

$$\mathscr{R}_{2,0}^{(4,+)} = -2^3(k_> + 1)(k_>^2 + 6k_> + 1)(4k_>^2 + 9k_> + 1)$$

$$\mathscr{R}_{1,1}^{(4,+)} = -2^4(k_> + 1)^2(3k_> + 1)(7k_>^2 + 12k_> + 1)$$

$$\mathscr{R}_{0,2}^{(4,+)} = -2^3(k_> + 1)^3(9k_>^3 + 27k_>^2 + 15k_> + 1)$$

$$\mathscr{R}_{0,0}^{(4,+)} = 3^2k_>^2(k_> + 1)^4$$

$T < T_c$:

$$\mathscr{R}_{4,0}^{(4,-)} = 2^4(49k_<^4 + 204k_<^3 + 86k_<^2 - 84k_< + 1)$$
$$\mathscr{R}_{3,1}^{(4,-)} = 2^6(k_< + 1)(5k_<^3 + 53k_<^2 + 71k_< - 1)$$
$$\mathscr{R}_{2,2}^{(4,-)} = 2^5(k_< + 1)(k_<^5 + 17k_<^4 - 20k_<^3 - 204k_<^2 - 181k_< + 3)$$
$$\mathscr{R}_{1,3}^{(4,-)} = -2^6(k_< + 1)^2(17k_<^3 + 61k_<^2 + 51k_< - 1)$$
$$\mathscr{R}_{0,4}^{(4,-)} = -2^4(k_< + 1)^3(9k_<^3 + 45k_<^2 + 43k_< - 1)$$
$$\mathscr{R}_{2,0}^{(4,-)} = -2^3k_<(k_< + 1)(k_<^2 + 6k_< + 1)(k_<^2 + 9k_< + 4)$$
$$\mathscr{R}_{1,1}^{(4,-)} = -2^6k_<(k_< + 1)^2(4k_<^2 + 3k_< + 1)$$
$$\mathscr{R}_{0,2}^{(4,-)} = -2^5k_<(k_< + 1)^3$$
$$\mathscr{R}_{0,0}^{(4,-)} = 3^2k_<^2(k_< + 1)^4$$

***n* = 5:**

$$\mathscr{R}_{5,0}^{(5)} = 2^7 \cdot 3(k^2 + 6k + 1)(k^4 - 132k^3 - 250k^2 - 132k + 1)$$
$$\mathscr{R}_{4,1}^{(5)} = 2^6(k^8 - 510k^7 - 6542k^6 - 9630k^5 + 14748k^4 + 39510k^3 + 24718k^2 + 3270k - 29)$$
$$\mathscr{R}_{3,2}^{(5)} = -2^9(k + 1)(15k^7 + 259k^6 - 135k^5 - 3687k^4 - 7335k^3 - 4803k^2 - 705k + 7)$$
$$\mathscr{R}_{2,3}^{(5)} = -2^6 \cdot 3(k + 1)^2(3k^7 + 72k^6 - 1095k^5 - 6878k^4 - 13515k^3 - 9756k^2 - 1617k + 18)$$
$$\mathscr{R}_{1,4}^{(5)} = 2^7(k + 1)^3(477k^5 + 2943k^4 + 6498k^3 + 5438k^2 + 1041k - 13)$$
$$\mathscr{R}_{0,5}^{(5)} = 2^6(k + 1)^4(3k + 1)(3k + 5)(9k^3 + 45k^2 + 75k - 1)$$
$$\mathscr{R}_{4,0}^{(5)} = 2^5(k^8 + 258k^7 + 904k^6 + 5310k^5 + 9582k^4 + 5310k^3 + 904k^2 + 258k + 1)$$
$$\mathscr{R}_{3,1}^{(5)} = 2^6(k + 1)(51k^7 + 398k^6 + 4131k^5 + 9582k^4 + 6489k^3 + 1410k^2 + 465k + 2)$$
$$\mathscr{R}_{2,2}^{(5)} = 2^5 \cdot 3(k + 1)^2(3k^7 + 42k^6 + 1443k^5 + 4502k^4 + 3801k^3 + 1054k^2 + 417k + 2)$$

$$\mathscr{R}_{1,3}^{(5)} = 2^7(k+1)^3(234k^5 + 981k^4 + 1044k^3 + 370k^2 + 186k + 1)$$

$$\mathscr{R}_{0,4}^{(5)} = 2^5(k+1)^4(81k^5 + 405k^4 + 522k^3 + 234k^2 + 165k + 1)$$

$$\mathscr{R}_{3,0}^{(5)} = 2^5 \cdot 3^2 k(k+1)^2(k^4 + 30k^3 + 74k^2 + 30k + 1)$$

$$\mathscr{R}_{2,1}^{(5)} = 2^4 \cdot 3^2 k(k+1)^2(k^6 + 13k^5 - 67k^4 - 358k^3 - 505k^2 - 167k - 5)$$

$$\mathscr{R}_{1,2}^{(5)} = -2^5 \cdot 3^2 k(k+1)^3(53k^4 + 179k^3 + 233k^2 + 77k + 2)$$

$$\mathscr{R}_{0,3}^{(5)} = -2^4 \cdot 3^2 k(k+1)^4(18k^4 + 81k^3 + 125k^2 + 47k + 1)$$

$$\mathscr{R}_{2,0}^{(5)} = -2^3 \cdot 3^2 k(k+1)^2(k^2 + 6k + 1)(k^4 + 15k^3 + 44k^2 + 15k + 1)$$

$$\mathscr{R}_{1,1}^{(5)} = -2^4 \cdot 3^2 k(k+1)^3(52k^4 + 147k^3 + 83k^2 + 21k + 1)$$

$$\mathscr{R}_{0,2}^{(5)} = -2^3 \cdot 3^2 k(k+1)^4(18k^4 + 63k^3 + 49k^2 + 21k + 1)$$

$$\mathscr{R}_{1,0}^{(5)} = 0$$

$$\mathscr{R}_{0,1}^{(5)} = 2^2 \cdot 3^4 k^3(k+1)^6$$

$$\mathscr{R}_{0,0}^{(5)} = 2 \cdot 3^4 k^3(k+1)^6$$

NOTE ADDED IN PROOF

After this paper was submitted for publication, H. Au-Yang and J. Perk submitted a paper in which the critical values $R_{n,c}$ are given for n up to 5.

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